OUTER MEASURE, BOREL SETS AND LEBESGUE MEASURE IN THE PLANE

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THESIS

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ABSTRACT

In this paper, the essential properties of general Lebesgue outer measure are discussed. The complete measure space, consisting of the general Lebesgue outer measure restricted to the measurable sets, is developed and this measure is shown to be unique. Two characterizations of measurable sets are discussed. The Borel sets are investigated in the plane and more generally, in n-space, and it is shown that the σ-algebra of Borel sets is equal to the product σ-algebra of Borel sets on the line. Finally, the interrelationships between Lebesgue measure in the plane and the product measure of Lebesgue measures on the line are investigated. It is shown that the σ-algebra of Lebesgue measurable sets properly contains the product σ-algebra and that these two measures agree on the product σ-algebra. It is also proven that the σ-algebra of Lebesgue measurable sets is the completion of the product σ-algebra. Examples are provided to illustrate that the product measure spaces discussed are not complete as well as an example of a subset of the plane which is not Lebesgue measurable.

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INTRODUCTION

The content of this thesis should be understandable to any reader whose background includes the equivalent of a course in measure and intergration theory. In fact, most of the questions answered in this paper were an outgrowth of a course in the theory of real variables taken by the author.

There are several ways of interrelating outer measure and measure. Some authors define a measure and extend it to an outer measure through hereditary sets. Others develop the notion of outer measure and restrict it to a certain class of sets to obtain a measure; this method will be discussed in this thesis.

In a first approach to measure and integration theory, attention is usually focused on Lebesgue measure and integration on the real line. An algebra is introduced as a collection of sets from a set, X, such that X and $\emptyset$ belong to this collection, and the collection is closed under complementation and finite unions. A σ-algebra is then defined to be an algebra which is closed under countable, vice finite, unions. The Borel sets are introduced as the smallest σ-algebra containing the open sets on the line. An outer measure is then defined, and this results in the definition of Lebesgue outer measure on the line. Using this outer measure, measurable sets are defined, and this

class of sets is found to be a σ-algebra. Then, equipped with this σ-algebra of measurable sets and Lebesgue outer measure (on the line), a measure space is formed; that is, Lebesgue outer measure restricted to be measurable sets is a measure and forms the measure space $(X,\mathcal{M},m)$. In fact, it is found that this measure space is complete and is the completion of the measure space $(X,\mathcal{B},m|_{\mathcal{B}})$, where $\mathcal{B}$ denotes the Borel sets on the real line.

Some natural questions then arise when we consider ways in which the theory for the real line can be extended to more general situations. In particular, what are the essential requirements in order that a set function may be used to define a general Lebesgue outer measure? Also, can Lebesgue outer measure always be used to define a collection of sets so that when the outer measure is restricted to these a complete measure space is obtained? These questions are answered in the first section.

There are two different approaches that may be taken in examining measure in higher dimensional spaces, based on a knowledge of measure on the line. The first is just the generalization of measure on the line by going directly to the entire space to define an outer measure and then inducing a measure. The second is through the use of product measures. Along with these measures the two σ-algebras are defined, one induced by the measure defined on the entire space and the other induced by the product measure. The question then arises: are these equivalent? Namely, are

their respective σ-algebras equal and are the measures of a set equal for all measurable sets? In particular, what happens in the Euclidean plane? By defining Lebesgue outer measure in the plane and restricting it to the measurable sets, a measure is obtained, namely Lebesgue measure in the plane. The relationship between this measure and the product measure of Lebesgue measure (on the line) crossed with itself is investigated. These measures are found to be not equivalent because the respective σ-algebras are not equal. Although this may be considered at first glance to be a defect in the theory, it does not constitute an essential problem because the one measure space is simply the completion of the other. Because of this fact, the theory of integration is unaffected since the integrals are the same using either measure, whenever the integrals have meaning. Some of these ideas are developed in the third section.

Through the examination of the above notions another question arises. Is the product σ-algebra of Borel sets equal to the σ-algebra of Borel sets in the higher dimensional space? This question arises due to the close association of Lebesgue measurable sets and Borel sets. The Lebesque measurable sets differ from the Borel sets only by a family of sets of measure zero. This question is answered in the second section and the answer is, interestingly enough, affirmative.

Notation in this thesis will follow normal conventions except as noted here. Define $\mathbb{P}^*$ to be the nonnegative part

of the extended real line. Also, use <> and {} to denote countable and arbitrary collections, respectively. Finally, let script capitals denote collections of sets, usually σ-algebras.

I. OUTER MEASURE

In this section, the notion of outer measure is formally introduced and examined for its essential properties. Measurable sets are defined and the properties of this class of sets are examined. It is shown that, in general, an outer measure restricted to the measurable sets yields a complete space. This restriction is also shown to be unique. Finally, a different view of a measurable set is examined.

An outer measure is a set function from the power set of a set, X, to the nonnegative, extended real line, $\mathbb{P}^*$, which is countably subadditive, monotone and such that the outer measure of the null set is zero.

Definition. Given a set, X, μ^* is an outer measure on X if $\mu^*: \wp(X) \to \mathbb{P}^*$ such that:

i) $\mu^*(\emptyset) = 0$

ii) $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B$.

iii) $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ for every sequence $\langle A_n \rangle_{n=1}^{\infty}$ from $\wp(X)$.

Based on the definition of Lebesgue outer measure on the real line, given a collection of sets $\mathcal{R}$ and a real-valued function r on $\mathcal{R}$, define $\mu^*: \wp(X) \to \mathbb{P}^*$ by $\mu^*(A) = \inf\{\Sigma r(R_i): \cup R_i \supseteq A, R_i \in \mathcal{R}\}$, and consider conditions on $\mathcal{R}$ and r, under which μ^* will define an outer measure.

Theorem 1.1. If

i) $\mathcal{R} \subseteq \mathcal{P}(X) \ni \bigcup_{i=1}^{\infty} R_i = X$

for some sequence $\langle R_i \rangle_{i=1}^{\infty}$ from $\mathcal{R}$,

ii) $r: \mathcal{R} \to \mathbb{P}^* \ni$:

a) if $R \subseteq S$ for $R, S \varepsilon \mathcal{R}$, then $r(R) \leq r(S)$,

b) $\forall\ \varepsilon > 0, \exists\ T \varepsilon \mathcal{R} \ni r(T) < \varepsilon$,

then μ^*, as defined above is an outer measure on the set X.

Proof.

i) By property b) of r, and since $\emptyset \subseteq R$, $\forall\ R \varepsilon \mathcal{R}$,

$$\mu^*(\emptyset) = \inf\{\Sigma\ r(R_i): \cup R_i \supseteq \emptyset,\ R_i \varepsilon \mathcal{R}\} = \inf\{r(R_i),\ R_i \varepsilon \mathcal{R}\} = 0.$$

$\therefore\ \mu^*(\emptyset) = 0.$

ii) Let $A \subseteq B \subseteq X$; then,

$$A \subseteq B \Rightarrow \{\Sigma\ r(R_i): \cup R_i \supseteq A\} \supseteq \{\Sigma\ r(R_i): \cup R_i \supseteq B\}$$

$$\Longrightarrow \inf\{\Sigma\ r(R_i): \cup R_i \supseteq A\} \leq \inf\{\Sigma\ r(R_i): \cup R_i \supseteq B\}$$

$\therefore\ \mu^*(A) \leq \mu^*(B).$

iii) Let $\langle A_i \rangle$ be a sequence from $\mathcal{P}(X)$.

Assume that $\mu^*(A_i) < \infty$, $\forall\ i$; for if $\mu^*(A_i) = \infty$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

follows immediately. Let $\varepsilon > 0$ and let $\{R_i^j\}$ be a countable collection from $\mathcal{R}$:

$$\bigcup_j R_i^j \supseteq A_i \quad \text{and} \quad \Sigma\ r(R_i^j) \leq \mu^*(A_i) + \varepsilon/2^i.$$

But

$$\bigcup_{i,j} R_i^j \supseteq \bigcup A_i \Rightarrow \inf\{\Sigma\, r(R_i) : \bigcup R_i \supseteq \bigcup A_i\} \leq \sum_{i,j} r(R_i^j).$$

$$\therefore\ \mu^*(\bigcup A_i) \leq \sum_{i,j} r(R_i^j) \leq \sum_i (\mu^*(A_i) + \varepsilon/2^i) \leq \sum_{i,j} \mu^*(A^i) + \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, hence

$\mu^*(\bigcup A_i) \leq \Sigma\, \mu^*(A_i)$ for $\{A_i\}$ a countable collection from $\mathcal{P}(X)$.

Hence μ^* is an outer measure.

Consider a subcollection of $\mathcal{P}(X)$, $\mathcal{A}$, namely the class of all subsets, A, of X such that $\forall E$, $E \in \mathcal{P}(X)$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E - A)$. $\mathcal{A}$ will be called the collection of all <u>μ^*-measurable</u> sets (also called the <u>admissible</u> sets) from $\mathcal{P}(X)$. This collection of admissible sets will turn out to form a σ-algebra, however there is no reason, in advance, to assume this fact. As a consequence of the manner in which μ^* was defined, μ^* is finitely additive on $\mathcal{A}$.

<u>Lemma 1.2</u>. Let μ^* be an outer measure on a set, X, and let $E \subseteq X$. If $\{A_i\}_{i=1}^n$ is any finite collection of pairwise disjoint admissible sets, then $\mu^*(\bigcup_{i=1}^n (A_i \cap E)) = \sum_{i=1}^n \mu^*(E \cap A_i)$.

<u>Proof</u>. Proceed by induction. For the case $n = 1$, the conclusion holds since the asserted equality reduces to the identity $\mu^*(E \cap A_1) = \mu^*(E \cap A_1)$. Now assume that the result holds for $n = k$ and let $\{A_i\}_{i=1}^{k+1}$ be a collection of $k + 1$ pairwise disjoint admissible sets. Then,

$$\mu^*(E \cap (\bigcup_{i=1}^{k+1} A_i)) = \mu^*((E \cap (\bigcup_{i=1}^{k+1} A_i)) \cap A_{k+1})$$

$$+ \mu^*((E \cap (\bigcup_{i=1}^{k+1} A_i)) - A_{k+1}).$$

But, since the A_i's are pairwise disjoint:

$$\mu^*(\bigcup_{i=1}^{k+1} (E \cap A_i)) = \mu^*(E \cap A_{k+1}) + \mu^*(E \cap (\bigcup_{i=1}^{k} A_i))$$

$$= \mu^*(E \cap A_{k+1}) + \mu^*(\bigcup_{i=1}^{k} (E \cap A_i))$$

$$= \mu^*(E \cap A_{k+1}) + \sum_{i=1}^{k} \mu^*(E \cap A_i)$$

$$\mu^*(\bigcup_{i=1}^{k+1} (E \cap A_i)) = \sum_{i=1}^{k+1} \mu^*(E \cap A_i).$$

<u>Corollary 1.3</u>. Let μ^* be an outer measure on a set, X, and $\{A_i\}_{i=1}^{k}$ be a collection of pairwise disjoint admissible sets from $\wp(X)$. Then $\mu^*(\bigcup A_i) = \sum \mu^*(A_i)$.

<u>Proof</u>. In the above lemma, let E = X.

The collection $\mathcal{A}$, of μ^*-measurable sets forms a σ-algebra, and when μ^* is restricted to this σ-algebra, μ^* satisfies the conditions of a measure (a nonnegative, countably additive set function, μ, on a σ-algebra, with values in $\mathbb{P}^*$). In fact, by passing to the measure space, $(X, \mathcal{A}, \mu^*|_{\mathcal{A}})$, in this manner we obtain a complete measure

space (i.e., every subset of a set of measure zero is measurable).

Lemma 1.4. Let μ^* be an outer measure on a set, X; then the collection, $\mathcal{A}$, of μ^*-measurable sets is an algebra.

Proof. In order to show that $\mathcal{A}$ is an algebra, it must be shown that:

i) $X, \emptyset \in \mathcal{A}$.

ii) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$.

iii) $\bigcup A_n \in \mathcal{A}$ whenever $\{A_n\}$ is a finite collection from $\mathcal{A}$.

i) $X, \emptyset \in \mathcal{A}$ by definition of μ^*-measurable sets

ii) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$, since $E \cap (X-A) = E - A$ and

$E - (X-A)) = E \cap A$.

that is,

$$A \in \mathcal{A} \Rightarrow \mu^*(E) = \mu^*(E \cap A) + \mu^*(E-A)$$
$$= \mu^*(E - (X-A)) + \mu^*(E \cap (X-A))$$

hence,

$X - A \in \mathcal{A}$.

iii) For $E \subseteq X$, it remains to show

$$\mu^*(E \cap (\bigcup_{i=1}^{n} A_i)) + \mu^*(E - (\bigcup_{i=1}^{n} A_i)) \leq \mu^*(E).$$

(Note: The reverse inequality is a direct consequence of the subadditivity of μ^*.)

$$\mu^*(E) = \mu^*(E \cap A_1) + \mu^*(E-A_1)$$

$$= \mu^*(E \cap A_1) + \mu^*((E-A_1) \cap A_2) + \mu^*((E-A_1)-A_2) + \dots$$

$$+ \mu^*((E-A_{n-1}) \cap A_n) + \mu^*((E-A_{n-1})-A_n).$$

$$= \mu^*(E \cap A_1) + \mu^*((E-A_1) \cap A_2) + \mu^*(E-(A_1 \cup A_2)) + \dots$$

$$+ \mu^*((E-A_{n-1}) \cap A_n)$$

$$+ \mu^*(E - (A_1 \cup A_2 \cup \dots \cup A_n)) .$$

But

$$(E \cap A_1) \cup ((E-A_1) \cap A_2) \cup \dots \cup ((E-A_{n-1}) \cap A_n)$$

$$= E \cap (A_1 \cup \dots \cup A_n).$$

$$\therefore \mu^*(E \cap A_1) + \mu^*((E-A_1) \cap A_2) + \dots$$

$$+ \mu^*((E-A_{n-1}) \cap A_n) \geq \mu^*(E \cap (A_1 \cup \dots \cup A_n)).$$

$$\therefore \mu^*(E) \geq \mu^*(E \cap (A_1 \cup \dots \cup A_n)) + \mu^*(E - (A_1 \cup \dots \cup A_n)).$$

$$\therefore \mu^*(E) = {}^*(E \cap (\bigcup_{i=1}^{n} A_i)) + \mu^*(E - (\bigcup_{i=1}^{n} A_i)).$$

Hence, $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ and thus, $\mathcal{A}$ is an algebra.

Theorem 1.5. $\mathcal{A}$ is a σ-algebra for a given set X.

Proof. Since it was just shown that $\mathcal{A}$ is an algebra, it remains to show that $\mathcal{A}$ is closed under countable unions.

Let $\langle A_n \rangle$ be a sequence of pairwise disjoint sets from $\mathcal{A}$, and set $C_n = \bigcup_{i=1}^{n} A_i$ and $A = \bigcup_{i=1}^{\infty} A_i$. For $E \subseteq X$, it must be shown that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E-A).$$

$$\forall n,\ C_n \in \mathcal{A} \Rightarrow \mu^*(E) = \mu^*(E \cap C_n) + \mu^*(E-C_n).$$

$$\therefore \forall n,\ \mu^*(E) \geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E-A).$$

Passing to the limit, we obtain

$$\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E-A)$$

$$\geq \mu^*\left(\bigcup_{i=1}^{\infty} (E \cap A_i)\right) + \mu^*(E-A)$$

$$\therefore\ \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E-A)$$

$\therefore\ \mathcal{A}$ is a σ-algebra.

Theorem 1.6. The restriction of μ^* to the admissible sets is a measure.

Proof. As a direct result of the definition of μ^*, $\mu^*(B) \geq 0$, $\forall\, B \subseteq X$, hence for the restriction of μ^* to be a measure, μ, it remains to show that μ is countably additive.

Let $\langle A_n \rangle$ be a sequence of pairwise disjoint admissible sets. Let $\mu^*|_{\mathcal{A}} = \mu$; then $\forall\, A \in \mathcal{A}$, $\mu(A) = \mu^*(A)$.

$$\therefore\ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

But,

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \mu^*(A_i) \leq \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right), \forall n.$$

Hence

$$\sum_{i=1}^{n} \mu(A_i) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right), \forall n.$$

Then by passage to the limit, $\sum_{i=1}^{\infty} \mu(A_i) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$.

$$\therefore\ \sum_{i=1}^{\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

Hence, $\mu^*|_{\mathcal{A}} = \mu$ is a measure.

Theorem 1.7. If μ is the restriction of μ^* to the admissible sets, then the resulting measure space $(X, \mathcal{A}, \mu)$ is complete.

Proof. To show $(X, \mathcal{A}, \mu)$ is complete it must be shown that every subset of a set of measure zero is measurable.

Let $A \subseteq X \ni: \mu^*(A) = 0$, then for $E \subseteq X$:

$$\mu^*(E) = \mu^*(E) + \mu^*(A) \geq \mu^*(E \cap A) + \mu^*(E - A).$$

$\therefore A \in \mathcal{A}$, i.e. every set of outer measure zero belongs to $\mathcal{A}$.

Now let $C \subseteq A$, then $\mu^*(C) = 0$.

$$\therefore C \in \mathcal{A}.$$

Hence $(X, \mathcal{A}, \mu)$ is complete.

Now that the essentials of Lebesgue outer measure, the induced Lebesgue measure, and some of their properties have been defined, we present a condition under which measures are equal. However, first we will recall the statement of the important Theorem on Monotone Classes: let X be a set and $\mathcal{C}$ a collection of subsets of X; if $\mathcal{A}$ is the algebra generated by $\mathcal{C}$, $\mathcal{B}$ the σ-algebra generated by $\mathcal{C}$, and $\mathcal{M}$ the monotone class generated by $\mathcal{A}$, then $\mathcal{B} = \mathcal{M}$.

Theorem 1.8.[1] Let $\bar{\mathcal{A}}$ be an algebra and suppose μ_1 and μ_2 are measures on a space X such that μ_1 and μ_2 are σ-finite and for all $A \in \bar{\mathcal{A}}$, $\mu_1(A) = \mu_2(A)$, then μ_1 and μ_2 agree on $\bar{\mathcal{A}}$, the σ-algebra generated by $\bar{\mathcal{A}}$.

Proof. Denote $\mathcal{A}^* = \{A \subseteq X: \mu_1(A) = \mu_2(A), A \in \mathcal{A}\}$.
First, assume that $\mu_1(X) = \mu_2(X) < \infty$.
Certainly, $\bar{\mathcal{A}} \subseteq \mathcal{A}^* \subseteq \mathcal{A}$ and it must be shown that $\mathcal{A} \subseteq \mathcal{A}^*$.
By the theorem on monotone classes, it suffices to show that $\mathcal{A}^*$ is a monotone class. Suppose $\langle E_n \rangle$ is a sequence from $\mathcal{A}^*$ such that $E_n \nearrow E$. Then since μ_1 and μ_2 are measures, for $i = 1,2$, $\mu_i(E_n) \nearrow \mu_i(E)$. But $\mu_1(E_n) = \mu_2(E_n)$, $\forall n$.
$\therefore$ $\mu_1(E) = \mu_2(E)$, since $\mu_i(E) = \lim_{n\to\infty} \mu_i(E_n)$, $i = 1,2$.
Hence $E \in \mathcal{A}$.

Similarly, if $\langle E_n \rangle$ is a sequence from $\mathcal{A}^* \ni E_n \searrow E$, then since $\mu_i(X) < \infty$, $\mu_i(E_n) \searrow \mu_i(E)$, $i = 1,2$, and as before
$\mu_1(E_n) = \mu_2(E_n)$, $\forall n$.
$\therefore$ $\mu_1(E) = \mu_2(E)$, since $\mu_i(E) = \lim_{n\to\infty} \mu_i(E_n)$, $i = 1,2$.
Hence $E \in \mathcal{A}$.

For X a finite measure space, $\mathcal{A}^*$ is a monotone class and $\mathcal{A}^* = \mathcal{A}$.

Now suppose that X is σ-finite and $E \in \mathcal{A}$. Let $\langle A_n \rangle$ be a sequence from $\bar{\mathcal{A}} \ni \bigcup_{n=1}^{\infty} A_n \supseteq E$ and $\mu_i(A_n) < \infty$, for $i = 1,2$ and all n.
Let $\bar{A}_n = \bigcup_{i=1}^{n} A_i$, then $\langle \bar{A}_n \rangle$ is a monotone increasing sequence.
$\therefore$ $\bar{A}_n \cap E \nearrow E \Rightarrow \mu_i(\bar{A}_n \cap E) \nearrow \mu_i(E)$, for $i = 1,2$.
Hence it suffices to show that $\mu_1(\bar{A}_n \cap E) = \mu_2(\bar{A}_n \cap E)$.
But $\mu_i(\bar{A}_n) < \infty$, for $i = 1,2$ and all n.
Then by the first half of the proof, $\mu_1(\bar{A}_n \cap E) = \mu_2(\bar{A}_n \cap E)$, $\forall n$.
$\therefore$ $\mu_1(E) = \mu_2(E)$, for $E \in \mathcal{A}$.

This theorem shows that we need only define the values which a measure assumes for sets which form an algebra and this guarantees equality on the generated σ-algebra. However, caution must be used when two measures are compared, as illustrated in the following.

Suppose μ and ν are measures on $[0,1] \ni : \mu([0,1]) = \nu([0,1]) = 1$ and $A_o, B_o \subseteq [0,1] \ni \mu(A_o) = \nu(B_o) = \frac{1}{2}$. Denote χ as the characteristic function of $(A_o \times Y)\Delta(X \times B_o)$ where $X = Y = [0,1]$ and let $f(x,y) = 2\chi(x,y)$. Define $\bar{\lambda}$ for all measurable sets $E \subseteq X \times Y$ (the unit square) to be:

$$\bar{\lambda}(E) = \int_E f(x,y)(\mu \times \nu).$$

Then $\bar{\lambda}(A \times Y) = \mu(A)$ and $\bar{\lambda}(X \times B) = \nu(B)$ for A, B measurable subsets of $[0,1]$. Hence $\bar{\lambda}$ and μ, and $\bar{\lambda}$ and ν agree on a certain class of sets but the class of sets is not an algebra, and it turns out that $\bar{\lambda}$, μ and ν are unique.[2]

In view of the above observations, another examination of conditions which yield a measurable set is warranted. Specifically, we ask if there is a relationship between outer measure and measurability of an unknown set other than the original definition of μ^*-measurability. The answer is a qualified yes; that is, there is a restriction on the type of measure space. Consider a set, X, which is equipped with a topology satisfying the second axiom of countability. Also, if a measure space is defined such that over the set X, a complete σ-finite measure space is formed, where the measure, μ, was constructed from an outer measure in which the underlying collection of sets, $\mathcal{R}$ (see Theorem 1.1), is

the topology, then measurability is equivalent to approximability from above and below by measurable sets.

Theorem 1.9. If X and the measure space on X are as prescribed above, then E is measurable if and only if for all $\varepsilon > 0$, there exists measurable sets E_1 and E_2 such that $E_1 \subseteq E \subseteq E_2$ and $\mu(E_2 - E_1) \leq \varepsilon$.[3]

Proof. Define F_σ and G_δ sets as countable unions of closed sets and countable intersections of open sets from X, respectively.

First, we assume that E is a measurable set. It will be shown that $\exists$ a G_δ-set, $0_\delta \ni \mu(0_\delta) = \mu(E)$ and $E \subseteq 0_\delta$.

Assume, first, that X is a finite measure space. Let $\langle 0_n \rangle_{n=1}^{\infty}$ be a sequence of open sets $\ni E \subseteq 0_n$, for all n and $\mu(E) \leq \mu(0_n) < \mu(E) + 1/n$. (Existence of this sequence is guaranteed by the manner in which μ was constructed.) Denote $\bigcap_{n=1}^{\infty} 0_n$ by 0_δ, then $E \subseteq 0_\delta \subseteq 0_n$, for all n and $\mu(E) \leq \mu(0_\delta) \leq \mu(0_n) < \mu(E) + 1/n$, for all n.

$\therefore \mu(E) = \mu(0_\delta)$.

Now assume that X is σ-finite. Then $\exists \langle E_n \rangle_{n=1}^{\infty} \ni E_n$ is measurable and has finite measure and $\bigcup_{n=1}^{\infty} E_n = E$.

Hence, for each E_n $\exists$ a sequence of open sets $\langle 0_i^n \rangle_{i=1}^{\infty} \ni E_n \subseteq 0_i^n$ for all n and i, and $\mu(0_i^n - E_n) \leq 1/(i \cdot 2^n)$.

Let $0_j = \bigcup_{n=1}^{\infty} 0_i^n$, then 0_j is an open set for all j and $E \subseteq 0_j$ and $\mu(0_j - E) \leq 1/j$.

Hence, $\bigcap_{j=1}^{\infty} 0_j$ is a G_δ-set $\ni \mu(\bigcap_{j=1}^{\infty} 0_j) = \mu(E)$ and $E \subseteq \bigcap_{j=1}^{\infty} 0_j$.

$\therefore$ For E measurable, $\exists$ a G_δ-set, 0_δ, $\ni$ $\mu(E) = \mu(0_\delta)$ and $0_\delta \supseteq E$.

It will now be shown that $\exists$ an F_σ-set, $F_\sigma \ni F_\sigma \subseteq E$ and $\mu(F_\sigma) = \mu(E)$. Let $\varepsilon > 0$ and consider $E' = X - E$.

From above, we know $\exists$ a G_δ-set, say $0^*, \ni 0^* \supseteq E'$ and $\mu(0^* - E') < \varepsilon$.

Then for $F = X - 0^*$, F is a G_δ-set $\ni F \subseteq E$.

But $E - F = E \cap (X-F) = E \cap 0^* = 0^* - (X-E) = 0^* - E'$.

$\therefore \mu(E-F) < \varepsilon$.

Therefore, if E is measurable, $\exists$ measurable sets E_1 and $E_2 \ni E_1 \subseteq E \subseteq E_2$ and $\mu(E_2-E_1) \leq \varepsilon$, for every $\varepsilon > 0$.

Finally, suppose that $\forall \varepsilon > 0$, $\exists$ measurable sets E_1 and $E_2 \ni E_1 \subseteq E \subseteq E_2$ and $\mu(E_2-E_1) \leq \varepsilon$. Thus, $\exists$ two sequences of measurable sets $\langle E_{1j} \rangle$ and $\langle E_{2j} \rangle \ni$: $E_{1j} \subseteq E \subseteq E_{2j}$ and $\mu(E_{2j}-E_{1j}) \leq 1/j$, $\forall j$.

Let $E_\sigma = \bigcup_{j=1}^{\infty} E_{1j}$ and $E_\delta = \bigcap_{j=1}^{\infty} E_{2j}$, then

$E_{1j} \subseteq E_\sigma \subseteq E \subseteq E_\delta \subseteq E_{2j}$, for all j.

$\therefore \mu(E_\delta - E_\sigma) = 0$.

But X is complete and $E_\delta - E \subseteq E_\delta - E_\sigma$.

$\therefore E_\delta - E$ is measurable.

Hence, E is measurable, since $E = E_\sigma \cup (E-E_\sigma)$.

II. BOREL SETS

Another σ-algebra is closely interrelated with the σ-algebra of Lebesgue measurable sets, namely the σ-algebra of Borel sets. The Lebesgue measurable sets and Borel sets differ only be sets of measure zero. In general, the σ-algebra of Borel sets is defined as the smallest σ-algebra containing the open sets of a topological space. Each Borel set can be approximated, to within a set of measure zero, by a countable union of open and closed sets, but the Borel sets remain elusive. In order to clarify this, consider the following classes of sets. A set which is a countable union of closed sets is called an F_σ, and a countable intersection of open sets is called a G_δ. Each F_σ and G_δ are Borel sets. Now consider the classes $F_{\sigma\delta}$ and $G_{\delta\sigma}$, where $F_{\sigma\delta}$ denotes countable intersections of F_σ's and $G_{\delta\sigma}$ denotes countable unions of G_δ's. As before, each $F_{\sigma\delta}$ and $G_{\delta\sigma}$ is a Borel set. Continuing in this manner yields two sequences of classes of sets: $F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, \ldots,$ and $G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, \ldots$. Now each set in each class is a Borel set but these sequences do not exhaust the σ-algebra of Borel sets. Hence a Borel set cannot, in general, be considered as a countable union or intersection of some combination of certain open or closed sets.

With this in mind, it seems apparent that when working with σ-algebras of Borel sets, a direct assault on the Borel sets through open sets must fail, and more subtle

approaches must be found. These approaches involve such notions as projection mappings and bases for a topology on a space. In particular, it is shown that the product σ-algebra of Borel sets on the line is equal to the σ-algebra of Borel sets in n-space. This is done first in the plane and then generalized to the finite dimensional case.

Lemma 2.1. If $\mathfrak{T}$ is the smallest σ-algebra containing the open sets of a topology on a space, X, which satisfies the second axiom of countability, then $\mathfrak{T}$ is the smallest σ-algebra containing any base, for the open sets.

Proof. Let $\mathfrak{T}'$ denote the σ-algebra generated by the basic open sets. Any basic open set is an open set, which implies $\mathfrak{T}$ contains the basic open sets.

$\therefore \mathfrak{T}' \subseteq \mathfrak{T}$.

Since X satisfies the second axiom of countability, every open set is the union of a countable collection of basic open sets and a countable union of basic open sets belongs to $\mathfrak{T}'$. Thus, every open set belongs to $\mathfrak{T}'$.

$\therefore \mathfrak{T} \subseteq \mathfrak{T}'$.

Hence, $\mathfrak{T} = \mathfrak{T}'$.

Corollary 2.2. $\mathcal{B}^2 \subseteq \mathcal{B} \times \mathcal{B}$.

Proof. $\mathcal{B}^2$ is the smallest σ-algebra containing the open sets in the plane. $\mathcal{B} \times \mathcal{B}$ is the smallest σ-algebra containing the Borel measurable rectangles. Since the collection of all open rectangles is a base for the open sets,

$\mathcal{B}^2$ is also the smallest σ-algebra containing the open rectangles. But every open rectangle is a Borel measurable rectangle and thus belongs to $\mathcal{B} \times \mathcal{B}$.

$\therefore \mathcal{B}^2 \subseteq \mathcal{B} \times \mathcal{B}$.

Theorem 2.3. $\mathcal{B} \times \mathcal{B}$ is the smallest σ-algebra in $\mathbb{R}^2$ such that the projection mappings, π_1 and π_2 ($\pi_1(x,y) = x$, $\pi_2(x,y) = y$) are measurable, i.e. $\pi_i^{-1}(B)$, $i = 1,2$ are measurable for $B \in \mathcal{B}$.

Proof. Denote the σ-algebra induced by π_1 and π_2 by $\mathcal{P}$. Let $B \in \mathcal{B}$ and consider $\pi_1^{-1}(B)$ and $\pi_2^{-1}(B)$.
$\pi_1^{-1}(B) = B \times \mathbb{R}$ and $\pi_2^{-1}(B) = \mathbb{R} \times B$ both of which are measurable rectangles and hence π_1^{-1} and π_2^{-1} preserve measurability relative to $\mathcal{B}$.
$\therefore \mathcal{P} \subseteq \mathcal{B} \times \mathcal{B}$, since $\mathcal{P}$ is the smallest σ-algebra with the property. It must now be shown that $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{P}$, i.e. every measurable rectangle belongs to $\mathcal{P}$.

Let $A \times C$ be a measurable rectangle, then $A \times C \in \mathcal{P}$, hence $A,C \in \mathcal{B}$. Then by the definition of $\mathcal{P}$, $\pi_i^{-1}(B)$, $i = 1,2$ are measurable for $B \in \mathcal{B}$, hence $\pi_1^{-1}(A) \in \mathcal{P}$ and $\pi_2^{-1}(C) \in \mathcal{P}$ and since $\mathcal{P}$ is a σ-algebra, $\pi_1^{-1}(A) \cap \pi_2^{-1}(C) \in \mathcal{P}$. But $\pi_1^{-1}(A) \cap \pi_2^{-1}(C) = (A \times \mathbb{R}) \cap (\mathbb{R} \times C) = A \times C \in \mathcal{P}$. Hence every measurable rectangle belongs to $\mathcal{P}$ and hence $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{P}$.
$\therefore \mathcal{B} \times \mathcal{B} = \mathcal{P}$.

Theorem 2.4. For the σ-algebra $\mathcal{B}^2$, π_1 and π_2 are measurable.

Proof. It must be shown that $\pi_1^{-1}(B) \in \mathcal{B}^2$ for $B \in \mathcal{B}$. It suffices to show that $\pi_1^{-1}(B) = B \times \mathbb{R} \in \mathcal{B}^2$.

Consider the following σ-algebras:

$\bar{\mathcal{B}} = \{B \times \mathbb{R},\ B \in \mathcal{B}\}$.

$\mathcal{J} = \{\pi_1^{-1}(B),\ B \in \mathcal{B}\}$, the σ-algebra induced by π_1.

$\mathcal{O} = \{\pi_1^{-1}(U),\ U \text{ open}\}$.

Then since $\pi_1^{-1}(B) = B \times \mathbb{R}$ for $B \not\in \mathcal{B}$, we have $\bar{\mathcal{B}} = \mathcal{J}$.

Also, $\mathcal{O} \subseteq \mathcal{J}$ since every open set is a Borel set.

Now, assume $\mathcal{O} \subset \mathcal{J}$, i.e. there exists a $B^* \in \mathcal{B} \ni B^* \times \mathbb{R} \not\in \mathcal{J}$ but $B^* \times \mathbb{R} \in \mathcal{O}$, and consider the σ-algebras $\mathcal{O}'$ and $\mathcal{J}'$ defined by:

$\mathcal{O}' = \{\pi_1(S),\ S \in \mathcal{O}\}$ and

$\mathcal{J}' = \{\pi_1(T),\ T \in \mathcal{J}\}$.

Then, since $B^* \times \mathbb{R} \not\in \mathcal{J}$ and $B^* \times \mathbb{R} \in \mathcal{O}$, $B^* \in \mathcal{O}'$ and $B^* \not\in \mathcal{J}'$, which is a contradiction since $\mathcal{O}' = \mathcal{J}'$, i.e. $\mathcal{J}'$ is the σ-algebra generated by the open sets on the line and $\mathcal{O}'$ is the σ-algebra of Borel sets on the line, hence $\mathcal{J}' = \mathcal{O}'$.

Then $\bar{\mathcal{B}} = \mathcal{O}$.

But $\pi_1^{-1}(U) = U \times \mathbb{R}$ which is an open rectangle for U an open set, hence $\pi_1^{-1}(U) \in \mathcal{B}^2$.

$\therefore \bar{\mathcal{B}} = \mathcal{O} \subseteq \mathcal{B}^2$.

$\therefore$ All rectangles of the form $B \times \mathbb{R}$ for $B \in \mathcal{B}$ belong to $\mathcal{B}^2$.

Then by symmetry, for $B \in \mathcal{B}$, $\mathbb{R} \times B \in \mathcal{B}^2$.

$\therefore$ For the σ-algebra $\mathcal{B}^2$, π_1 and π_2 preserve measurability relative to $\mathcal{B}$.

Corollary 2.5. $\mathcal{B}^2 = \mathcal{B} \times \mathcal{B}$.

Proof. $\mathcal{B} \times \mathcal{B}$ is the smallest σ-algebra such that π_1 and π_2 preserve measure and for $\mathcal{B}^2$, π_1 and π_2 preserve measure, hence $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{B}^2$.

But by corollary 2.2, $\mathcal{B}^2 \subseteq \mathcal{B} \times \mathcal{B}$.

$\therefore \mathcal{B}^2 = \mathcal{B} \times \mathcal{B}$.

In order to facilitate the generalizations of the above proofs to n-space, the following notation will be used. Let $\mathcal{B} \times^n \mathcal{B}$ denote $\mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_n$ where $\mathcal{B}_i = \mathcal{B}$, the σ-algebra of Borel sets. That is, $\mathcal{B} \times^n \mathcal{B}$ is the smallest σ-algebra containing the (Borel) measurable n-rectangles.

Theorem 2.6. In $\mathbb{R}^n$, $\mathcal{B}^n = \mathcal{B} \times^n \mathcal{B}$.

Proof. $\mathcal{B} \times^n \mathcal{B}$ is the smallest σ-algebra containing the measurable rectangles. $\mathcal{B}^n$ is the smallest σ-algebra containing the open sets in $\mathbb{R}^n$. Since $\mathbb{R}^n$ satisfies the second axiom of countability, by lemma 2.1, $\mathcal{B}^n$ is also the smallest σ-algebra containing the open n-rectangles.

We first prove that $\mathcal{B}^n \subseteq \mathcal{B} \times^n \mathcal{B}$. Every open n-rectangle is a measurable n-rectangle and hence belongs to $\mathcal{B} \times^n \mathcal{B}$, hence $\mathcal{B}^n \subseteq \mathcal{B} \times^n \mathcal{B}$.

We next show $\mathcal{B} \times^n \mathcal{B}$ is the smallest σ-algebra for which the projection mappings, $\pi_i(x_1, x_2, \ldots, x_n) = x_i$ for $i = 1, 2, \ldots, n$, are measurable relative to $\mathcal{B}$. Denote this smallest σ-algebra by $\mathcal{P}^n$ and it remains to show that $\mathcal{B} \times^n \mathcal{B} = \mathcal{P}^n$.

Note that $\pi_i^{-1}(B) = \mathbb{R}_1 \times \ldots \times \mathbb{R}_{i-1} \times B \times \mathbb{R}_{i+1} \times \ldots \times \mathbb{R}_n$, for $i = 1, 2, \ldots, n$, where $B \in \mathcal{B}$ and $\mathbb{R}_i = \mathbb{R}$ for all i.

$\pi_i^{-1}(B)$, $i = 1, 2, \ldots, n$ is a measurable rectangle and hence for $\mathcal{B} \times^n \mathcal{B}$, π_i is Borel measurable.

$\therefore \mathcal{P}^n \subseteq \mathcal{B} \times^n \mathcal{B}$.

Let $A = A_1 \times \ldots \times A_n \in \mathcal{B} \times^n \mathcal{B}$ and consider $\pi_i^{-1}(A_i)$.

Since, for $\mathcal{P}^n$, π_i is Borel measurable, $\pi_i^{-1}(A_i) \in \mathcal{P}^n$ for $i = 1, 2, \ldots, n$.

But $\mathcal{P}^n$ is a σ-algebra, hence $\bigcap_{i=1}^{n} \pi_i^{-1}(A_i) \in \mathcal{P}^n$.

But $\bigcap_{i=1}^{n} \pi_i^{-1}(A_i) = A$, hence $A \in \mathcal{P}^n$.

$\therefore \mathcal{B} \times^n \mathcal{B} \subseteq \mathcal{P}^n$.

$\therefore \mathcal{B} \times^n \mathcal{B} = \mathcal{P}^n$.

It remains to show that for $\mathcal{B}^n$, π_i is Borel measurable. It suffices to show that $\pi_1^{-1}(B) = B \times \mathbb{R}_2 \times \ldots \times \mathbb{R}_n \in \mathcal{B}^n$, for then by symmetry $\pi_i^{-1}(B) \in \mathcal{B}^n$. Hence, since $\mathcal{B} \times^n \mathcal{B}$ is the smallest σ-algebra such that the projection mappings are Borel measurable, it will follow that $\mathcal{B} \times^n \mathcal{B} \subseteq \mathcal{B}^n$.

Consider the following σ-algebras:

$$\bar{\mathcal{S}} = \{\pi_1^{-1}(B),\ B \in \mathcal{B}\}.$$

$$\bar{\mathcal{O}} = \{\pi_1^{-1}(U),\ U \text{ open}\}.$$

$$\bar{\mathcal{B}} = \{B \times \mathbb{R}_2 \times \ldots \times \mathbb{R}_n = B \times \mathbb{R}^{n-1},\ B \in \mathcal{B}\}.$$

Then, since $\pi_1^{-1}(B) = B \times \mathbb{R}^{n-1}$ for $B \in \mathcal{B}$, $\mathcal{B}^* = \bar{\mathcal{J}}$. Also, $\bar{\mathcal{O}} \subseteq \bar{\mathcal{J}}$ since every open set is a Borel set. Now, assume $\bar{\mathcal{O}} \subset \bar{\mathcal{J}}$, i.e. $\exists\ \tilde{B} \times \mathbb{R}^{n-1} \notin \bar{\mathcal{J}} \ni \tilde{B} \times \mathbb{R}^{n-1} \in \bar{\mathcal{O}}$ and consider the following two σ-algebras:

$$\bar{\mathcal{O}}' = \{\pi_1(S),\ S \in \bar{\mathcal{O}}\} \text{ and}$$

$$\bar{\mathcal{J}}' = \{\pi_1(T),\ T \in \bar{\mathcal{J}}\}.$$

Then $\tilde{B} \times \mathbb{R}^{n-1} \notin \bar{\mathcal{J}}$ and $\tilde{B} \times R^{n-1} \in \bar{\mathcal{O}}$ implies that $\tilde{B} \in \bar{\mathcal{O}}'$ and $\tilde{B} \notin \bar{\mathcal{J}}'$ which is a contradiction since $\bar{\mathcal{J}}' = \bar{\mathcal{O}}'$.
($\bar{\mathcal{J}}'$ is the σ-algebra generated by the open sets which is, in fact, the σ-algebra of Borel sets, namely, $\bar{\mathcal{O}}'$.)

$\therefore\ \mathcal{B} \times^n \mathcal{B} \subseteq \mathcal{B}^n$.

Hence $\mathcal{B} \times^n \mathcal{B} = \mathcal{B}^n$.

<u>Theorem 2.7</u>. $\mathcal{B} \times^n \mathcal{B} = \mathcal{B}^n$ iff $\mathcal{B}^s \times \mathcal{B}^t = \mathcal{B}^{s+t}$ for $n,s,t < \infty$.

<u>Proof</u>. Suppose $\mathcal{B} \times^n \mathcal{B} = \mathcal{B}^n$ for all finite n.

Consider $n = s$ and $n = t$; then $\mathcal{B}^s = \mathcal{B} \times^s \mathcal{B}$ and $\mathcal{B}^t = \mathcal{B} \times^t \mathcal{B}$.

But $\mathcal{B}^s = \mathcal{B}_1 \times \ldots \times \mathcal{B}_s$ and $\mathcal{B}^t = \mathcal{B}_1 \times \ldots \times \mathcal{B}_t$,

hence $\mathcal{B} \times^{s+t} \mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_{s+t} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_s \times \mathcal{B}_{s+1} \times \ldots \times \mathcal{B}_{s+t}$.

$\therefore\ \mathcal{B} \times^{s+t} \mathcal{B} = \mathcal{B}^s \times \mathcal{B}^t$.

But $\mathcal{B} \times^{s+t} \mathcal{B} = \mathcal{B}^{s+t}$.

$\therefore\ \mathcal{B}^s \times \mathcal{B}^t = \mathcal{B}^{s+t}$.

Now suppose that $\mathcal{B}^s \times \mathcal{B}^t = \mathcal{B}^{s+t}$ for all s,t finite. Then, in particular, $\mathcal{B}^n \times \mathcal{B} = \mathcal{B}^{n+1}$ for all finite n

and by applying this to $\mathcal{B} \times^n \mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ we obtain $\mathcal{B}^n$.

i.e. $\mathcal{B} \times^n \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_n = \mathcal{B}^2 \times \mathcal{B}_3 \times \ldots \times \mathcal{B}_n = \mathcal{B}^3 \times \mathcal{B}_4 \times \ldots \times \mathcal{B}_n = \ldots = \mathcal{B}^{n-1} \times \mathcal{B} = \mathcal{B}^n$.

$\therefore \mathcal{B} \times^n \mathcal{B} = \mathcal{B}^n$.

III. LEBESGUE MEASURE IN THE PLANE

With the notions of the general Lebesgue measure and those of Borel sets developed, we now turn to the Euclidean plane and investigate Lebesgue measure in the plane and its relation to the product measure of Lebesgue measure on the line. We will construct Lebesgue outer measure in the plane and then consider the measurable sets with regard to this outer measure to form the complete measure space $(\mathbb{R}^2, \mathcal{M}^2, m_2)$. It will be shown that the σ-algebra of Lebesgue measurable sets in the plane properly contain the σ-algebra generated by the Lebesgue measurable rectangles. Also, it will be shown that, on the σ-algebra generated by the Lebesgue measurable rectangles, the measure induced in the plane and the product measure agree. This measure space, $(\mathbb{R}^2, \mathcal{M}^2, m_2)$, will turn out to be the completion of the product measure space as well as the completion of the measure space $(\mathbb{R}^2, \mathcal{B}^2, m_2|_{\mathcal{B}^2})$, where $\mathcal{B}^2$ denotes the Borel sets in the plane.

In order to facilitate understanding of this section, we will review some of the notions involved with product measures. Consider the sets E_x and E^y defined, for E a measurable set, by $E_x = \{y \in \mathbb{R}: (x,y) \in E\}$ and $E^y = \{x \in \mathbb{R}: (x,y) \in E\}$. These will be referred to as the <u>x- and y-sections of E</u> or, more generally, as the <u>cross-sections of E</u>. Now consider the functions $\Phi(x)$ and $\Psi(y)$ defined on the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$, to be:

$\Phi(x) = m(E_x)$ and $\Psi(y) = m(E^y)$. Finally, integration of these functions over the entire real line yields $\int_{\mathbb{R}} \Phi(x)\, dm = \int_{\mathbb{R}} \Psi(y)\, dm$; this common value will be known as the <u>product measure of E</u>, and denoted $(m \times m)E$.

Let $\mathcal{R}$ denote the set of all open rectangles in $\mathbb{R}^2$. Then for $R \in \mathcal{R}$, R is of the form $(a_1,a_2) \times (b_1,b_2)$. Define the function r as follows:

$r\colon \mathcal{R} \to \mathbb{P}^*$ such that, for $R = (a_1,a_2) \times (b_1,b_2)$, $r(R) = (a_2-a_1)(b_2-b_1)$, i.e. $r(R)$ is the area of the rectangle R. Then by theorem 1.1, we can define the outer measure, m_2^*, by: For $A \subseteq \mathbb{R}^2$,

$$m_2^*(A) = \inf\{\Sigma r(R_i)\colon \bigcup R_i \supseteq A \text{ and } R_i \in \mathcal{R} \text{ for all } i\}.$$

Hence we generate the complete measure space $(R^2, \mathcal{M}^2, m_2)$ where $\mathcal{M}^2$ is the σ-algebra defined by:

$A \in \mathcal{M}^2$ if and only if for all $E \subseteq \mathbb{R}^2$, $m_2^*(E) = m_2^*(E \cap A) + m_2^*(E - A)$.

Then by lemma 1.4 and theorem 1.5, $\mathcal{M}^2$ is the σ-algebra of Lebesgue measurable sets in the plane. Also, m_2 denotes the Lebesgue outer measure m_2^* restricted to the measurable sets. By theorem 1.6, m_2 is a measure and the measure space $(R^2, \mathcal{M}^2, m_2)$ is complete by theorem 1.7.

<u>Theorem 3.1</u>. $(\mathbb{R}^2, \mathcal{M}^2, m_2)$ is the completion of $(\mathbb{R}^2, \mathcal{B}^2, m_2')$, where $m_2' = m_2|_{\mathcal{B}^2}$.

<u>Proof</u>. Note that $(R^2, \mathcal{M}^2, m_2)$ is complete and $\mathcal{B}^2 \subseteq \mathcal{M}^2$ since $\mathcal{M}^2$ contains all measurable rectangles and hence contains all open sets in $\mathbb{R}^2$. Therefore it suffices to show that if

$A \in \mathcal{M}^2$, then A can be expressed as $A = B \cup C$ where $B \in \mathcal{B}^2$ and $m_2(C) = 0$.

Since $A \in \mathcal{M}^2$, by theorem 1.9, there exists a sequence of measurable sets $\langle B_n \rangle$ such that $B_n \subseteq A$ and $B_n \subseteq B_{n+1}$ for all n and further, $m_2'(B_n) = m_2(B_n) = m_2(A) - 1/n$, for all n.

Put $B = \bigcup_{n=1}^{\infty} B_n$ and $C = A - B$.

Now, $\bigcup_{n=1}^{\infty} B_n \subseteq A$, hence $m_2(B) \leq m_2(A)$.

But $m_2(B) = \lim_{n\to\infty} m_2(B_n) \geq m_2(A)$.

$\therefore$ $m_2(B) = m_2(A)$

Hence, $m_2(C) = 0$.

Further, since $B_n \in \mathcal{B}^2$ for all n, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}^2$.

$\therefore$ $(\mathbb{R}^2, \mathcal{M}^2, m_2)$ is the completion of $(\mathbb{R}^2, \mathcal{B}^2, m_2')$.

Lemma 3.2. $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{M} \times \mathcal{M}$.

Proof. $\mathcal{B} \times \mathcal{B}$ is the smallest σ-algebra containing the Borel measurable rectangles and $\mathcal{M} \times \mathcal{M}$ is the smallest σ-algebra containing the Lebesgue measurable rectangles. But, each Borel rectangle is a Lebesgue rectangle. Hence, $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{M} \times \mathcal{M}$.

Lemma 3.3. $\mathcal{B} \times \mathcal{B}$ is not complete.

Proof. Let C_1 and $C_2 \in \mathcal{M}$ such that $m(C_1) = m(C_2) = 0$ and $C_1, C_2 \notin \mathcal{B}$, and consider $C_1 \times C_2$. Assume $C_1 \times C_2 \in \mathcal{B} \times \mathcal{B}$, then $C_1 \times C_2$ would be a Borel measurable rectangle, which is a contradiction because each cross-section of $C_1 \times C_2$

would have to be measurable and neither C_1 nor C_2 are Borel measurable. Hence $\mathcal{B} \times \mathcal{B}$ does not contain all subsets of measure zero and therefore $\mathcal{B} \times \mathcal{B}$ is not complete.

Theorem 3.4. $\mathcal{M} \times \mathcal{M} \subseteq \mathcal{M}^2$.

Proof. In order to show that $\mathcal{M} \times \mathcal{M} \subseteq \mathcal{M}^2$, it suffices to show that each measurable rectangle belongs to $\mathcal{M}^2$. Since every measurable set is the union of a Borel set and a set of measure zero, each measurable rectangle, $M \times N$, can be considered as the union of four parts, that is for $B_1, B_2 \in \mathcal{B}$ and C_1, C_2 sets of measure zero, $M \times N =$ $(B_1 \times B_2) \cup (C_1 \times B_2) \cup (B_1 \times C_2) \cup (C_1 \times C_2)$.

Since $\mathcal{B}^2 \subseteq \mathcal{M}^2$ and $\mathcal{B} \times \mathcal{B} = \mathcal{B}^2$, for $B_1, B_2 \in \mathcal{B}$, $B_1 \times B_2 \in \mathcal{M}^2$, and therefore it remains to be shown that rectangles of the form $B \times C$ and $C_1 \times C_2$ belong to $\mathcal{M}^2$. $(R^2, \mathcal{M}^2, m_2)$ is complete, hence it contains all sets and subsets of measure zero. Therefore, if $m_2^*(B \times C) =$ $m_2^*(C_1 \times C_2) = 0$, then $B \times C$ and $C_1 \times C_2$ belong to $\mathcal{M}^2$.

Suppose $B \in \mathcal{B} \ni B \subseteq I^*$, where $\ell(I^*) = b - a < \infty$, and $C \in \mathcal{M} \ni m(C) = 0$.

Then:

$$m_2^*(B \times C) = \inf\{\Sigma r(R_i): \bigcup R_i \supseteq B \times C\} \geq 0,$$

$$m(C) = \inf\{\Sigma \ell(I_i): \bigcup I_i \supseteq C\} = 0,$$

$$m(B) = \inf\{\Sigma \ell(I_i): \bigcup I_i \supseteq B\} < \infty.$$

$m(C) = 0 \Rightarrow \forall\, \varepsilon > 0,\ \exists\ \langle \bar{I}_i \rangle \ni \bigcup \bar{I}_i \supseteq C$ and $\Sigma \ell(\bar{I}_i) \leq \varepsilon / b-a$. Let $\bar{R}_i = I^* \times \bar{I}_i$, then $\bigcup \bar{R}_i = \bigcup (I^* \times \bar{I}_i) \supseteq B \times C$ and

$\Sigma r(\overline{R}_i) = \Sigma r(I^* \times \overline{I}_i) = \Sigma \ell(I^*)\ell(\overline{I}_i) = \ell(I^*)\Sigma\ell(\overline{I}_i) \leq \varepsilon.$

Hence, $m_2^*(B \times C) \leq \varepsilon$, $\forall \varepsilon > 0$ and $m_2^*(B \times C) \geq 0$.

$\therefore$ $m_2^*(B \times C) = 0$.

Now let $B \varepsilon \mathcal{B}$ be arbitrary; then $B \subseteq (-\infty,\infty)$, and

$\exists$ $\langle B_j \rangle \ni \bigcup B_j = B$ and $\forall j$, $B_j \subseteq I_j$ where $\ell(I_j) = b_j - a_j < \infty$.

Then $(\bigcup B_j) \times C = B \times C = \bigcup(B_j \times C)$.

Thus, $m_2^*(\bigcup(B_j \times C)) \leq \Sigma m_2^*(B_j \times C)$. But for all j,

$m_2^*(B_j \times C) = 0$.

$\therefore$ $m_2^*(B \times C) = 0$, for $B \varepsilon \mathcal{B}$.

Now, let $C_1, C_2 \varepsilon \mathfrak{M}$ where $m(C_1) = m(C_2) = 0$.

Then $C_1 \times C_2 \subseteq \mathbb{R} \times C_2$, hence $m_2^*(C_1 \times C_2) \leq m_2^*(\mathbb{R} \times C_2)$

But $m_2^*(\mathbb{R} \times C_2) = 0$

$\therefore$ $m_2^*(C_1 \times C_2) = 0$.

By symmetry, $m_2^*(C \times B) = 0$ follows and hence every measurable rectangle belongs to $\mathfrak{M}^2$.

$\therefore$ $\mathfrak{M} \times \mathfrak{M} \subseteq \mathfrak{M}^2$.

<u>Theorem 3.5</u>. If $M \subseteq \mathbb{R}^2$ and $M \varepsilon \mathfrak{M} \times \mathfrak{M}$, then $(m \times m)M = m_2 M$.

<u>Proof</u>. Assume that $M \subseteq I_1 \times I_2$ where I_1 and I_2 are finite intervals. Then $(m \times m)M = \int_{\mathbb{R}} \Phi(x)dm = \int_{\mathbb{R}} m(M_x)dm$ and Φ is a nonnegative measurable function; further, since $M \subseteq I_1 \times I_2$, Φ is bounded and defined on a bounded interval. Hence there exists a sequence of simple functions (a real valued function whose range consists of a finite set of distinct points), $\langle t_n \rangle$ such that $t_n \searrow \Phi$; further $\int_{\mathbb{R}} t_1 dm < \infty$, therefore by the dominated convergence theorem, $\lim_{n\to\infty} \int_{\mathbb{R}} t_n dm = \int_{\mathbb{R}} \Phi\, dm$. (The

dominated convergence theorem states that if a sequence $<f_n>$ of measurable functions and $\lim_{n\to\infty} f_n = f$, for f a measurable function, and if there exists an integrable function g such that $|f_n| \leq g$ for all n, then $\lim_{n\to\infty} \int f_n dm = \int f dm$.) Each t_n defines a set of rectangles covering the area under the graph of the cross-section functions and by translating these properly will yield a sequence of rectangels, $<T_n>$, covering M. Let $\varepsilon > 0$, let $\overline{t}_n$ be the simple function such that $\int_{\mathbb{R}} \Phi\, dm + \varepsilon \geq \int_{\mathbb{R}} \overline{t}_n dm$, and let $<\overline{T}_n>$ by the sequence of rectangles covering M associated with $\overline{t}_n$, i.e. $\bigcup \overline{T}_n \supseteq M$ and $\Sigma r(\overline{T}_n) = \int_{\mathbb{R}} \overline{t}_n dm$.

Thus $\inf \{\Sigma r(T_n): \bigcup T_n \supseteq M\} \leq \Sigma r(\overline{T}_n) \leq \int_{\mathbb{R}} \Phi\, dm + \varepsilon$.

Hence $m_2 M \leq (m \times m)M$.

Now, let $<s_n>$ be a sequence of simple functions such that $s_n \nearrow \Phi$. Then by the monotone convergence theorem, $\lim_{n\to\infty} \int_{\mathbb{R}} s_n dm = \int_{\mathbb{R}} \Phi\, dm$. (The monotone convergence theorem states that if a sequence $<f_n>$ of non-negative measurable functions is such that $f_n \nearrow f$ for f a measurable function, then $\lim_{n\to\infty} \int f_n dm = \int f\, dm$.). As above, each s_n defines a sequence of rectangles covered by M, $<S_n>$. Let $\varepsilon > 0$, let $\overline{s}_n$ be the simple function such that $\int_{\mathbb{R}} \Phi dm - \varepsilon \leq \int_{\mathbb{R}} \overline{s}_n dm$, and let $<\overline{S}_n>$ denote the associated sequence of rectangles, i.e. $\bigcup \overline{S}_n \subseteq M$ and $\Sigma r(\overline{S}_n) = \int_{\mathbb{R}} \overline{s}_n dm$.

Then $\inf\{\Sigma r(S_n): \bigcup S_n \subseteq M\} \geq \Sigma r(\overline{S}_n) \geq \int_{\mathbb{R}} \Phi\, dm - \varepsilon$.

Hence, $m_2 M \geq (m \times m)M$.

$\therefore$ $m_2 M = (m \times m)M$, for M a subset of bounded rectangle.

Now let $M \in \mathfrak{M} \times \mathfrak{M}$, M arbitrary. Since both $(\mathbb{R}^2, \mathfrak{M} \times \mathfrak{M}, m \times m)$ and $(\mathbb{R}^2, \mathfrak{M}^2, m_2)$ are σ-finite, there exists a sequence $\langle R_i \rangle$ of rectangles such that $\bigcup R_i = \mathbb{R}^2$ and R_i is a subset of bounded rectangle for every i.

Then $M = M \cap (\bigcup_{i=1}^{\infty} R_i) = \bigcup_{i=1}^{\infty} (M \cap R_i)$ and

$$m_2(M) = m_2(\bigcup_{i=1}^{\infty} (M \cap R_i)) = \sum_{i=1}^{\infty} m_2(M \cap R_i)$$

$$= \sum_{i=1}^{\infty} (m \times m)(M \cap R_i) = (m \times m)(\bigcup_{i=1}^{\infty} (M \cap R_i))$$

$$= (m \times m)M.$$

$\therefore\ m_2 M = (m \times m)M$ for $M \in \mathfrak{M} \times \mathfrak{M}$.

<u>Theorem 3.6</u>. $(\mathbb{R}^2, \mathfrak{M}^2, m_2)$ is the completion of $(\mathbb{R}^2, \mathfrak{M} \times \mathfrak{M}, m \times m)$.

<u>Proof</u>. It suffices to show that for $A \in \mathfrak{M}^2$, A can be expressed as $A = B \cup C$ where $B \in \mathfrak{M} \times \mathfrak{M}$ and $m_2(C) = 0$. But $(\mathbb{R}^2, \mathfrak{M}^2, m_2)$ is the completion of $(\mathbb{R}^2, \mathfrak{B}^2, m_2|_{\mathfrak{B}^2})$, so A can be expressed as $A = B \cup C$ where $B \in \mathfrak{B}^2$ and $m_2(C) = 0$. However, from Corollary 2.5, $\mathfrak{B} \times \mathfrak{B} = \mathfrak{B}^2$, hence we have $\mathfrak{B}^2 \subseteq \mathfrak{M} \times \mathfrak{M}$. Also, from the preceeding theorem we have $m_2|_{\mathfrak{M} \times \mathfrak{M}} = m \times m$. Hence $(\mathbb{R}^2, \mathfrak{M}^2, m_2)$ is the completion of $(\mathbb{R}^2, \mathfrak{M} \times \mathfrak{M}, m \times m)$.

<u>Lemma 3.7</u>. Let f be a Borel measurable function and denote the graph of f by L; then $m_2(L) = 0$.

Proof. Consider the cross-section function $\Psi(y) = m(L^y)$. Since L is the graph of a function L^y is a single point for all y, hence $\Psi(y) = 0$ for all y. Thus $(m \times m)L = \int_{\mathbb{R}} \Psi(y)\,dm = \int_{\mathbb{R}} 0\,dm = 0$. But, since $L \in \mathcal{M} \times \mathcal{M}$, $m_2 L = (m \times m)L = 0$.

$\therefore m_2 L = 0$.

It should be noted that in the above lemma, the term "function" was used in its strictest sense. The conclusion of the lemma is valid also for countable unions of graphs of measurable functions, when either variable is taken to be independent. For example, the graph of the equation x = constant has measure zero. However, for more general f, $f: [0,1] \to \mathbb{R}^2$, a measurable mapping, the range of f need not have measure zero. One example is the space filling curve.[4] This curve is a continuous, hence measurable, mapping of $[0,1]$ onto the square $[0,1] \times [0,1]$ and $m_2([0,1] \times [0,1]) = 1 > 0$.

The following theorem is an immediate consequence of theorem 3.5. We offer an independent proof since it may be possible to deduce theorem 3.5 from this theorem. It must, however, be shown that if the measure of the rectangle is equal to its area, then $m \times m$ and m_2 agree on the measurable rectangles. Once this is shown, theorem 3.5 can be obtained from theorem 3.8 by application of the theorem on monotone classes and the σ-finiteness of the measure space.

Theorem 3.8. The measure of an arbitrary rectangle is equal to its area.

Proof. Let R denote some rectangle, then $m_2(R) = \inf\{\Sigma r(R_i): \bigcup R_i \supseteq R\} \leq A$ where A denotes the area of R, since $R \supseteq R$ and $r(R) = A$.

Hence it remains to be shown that $m_2(R) \leq A$.

First, suppose R is compact, i.e. R is closed and bounded. Then $R = I_1 \times I_2$ where $I_1 = [a_1,b_1]$ and $I_2 = [a_2,b_2]$ and $-\infty < a_1,a_2,b_1,b_2 < \infty$. Since I_1 and I_2 are compact, there exist finite subcovers $\langle I_{1i}\rangle$ and $\langle I_{2j}\rangle$ for any open cover of I_1 and I_2. Let $\langle I^*_{1i}\rangle$ and $\langle I^*_{2j}\rangle$ be open covers for I_1 and I_2 such that for $I^*_{1i} \times I^*_{2j} = R_{ij}$, $\sum_{i,j} r(R_{ij}) \leq m_2(R) + \varepsilon$ for some $\varepsilon > 0$. Now for the respective finite subcovers $\langle I_{1i}\rangle$ and $\langle I_{2j}\rangle$ define $I_{1i} \times I_{2j} = \overline{R}_{ij}$.

Then $\bigcup_{i,j} \overline{R}_{ij} \subseteq \bigcup_{i,j} R_{ij}$ and hence $\sum_{i,j} r(\overline{R}_{ij}) \leq \sum_{i,j} r(R_{ij}) \leq m_2(R) + \varepsilon$.

Now without loss of generality, arrange the finite subcovers of I_1 and I_2 such that for $I_{1i} = (a_{1i},b_{1i})$ and $I_{2j} = (a_{2j},b_{2j})$ we have:

$$\begin{array}{ccc} a_{11} < a_1 < b_{11} & & a_{21} < a_2 < b_{21} \\ a_{12} < b_{11} < b_{12} & \text{and} & a_{22} < b_{21} < b_{22} \\ \vdots & & \vdots \\ a_{1n} < b_1 < b_{1n} & & a_{2n} < b_2 < b_{2n} \end{array}$$

Then $\overline{R}_{ij} = I_{1i} \times I_{2j} = (a_{1i},b_{1i})\,(a_{2j},b_{2j})$ and $\bigcup_{i,j} \overline{R}_{ij} = (\bigcup_i I_{1i}) \times (\bigcup_j I_{2j}) = \bigcup_{i,j} (I_{1i} \times I_{2j}) \supseteq R$.

Hence, $r(\bigcup_{i,j} \bar{R}_{ij}) = r(\bigcup_{i,j} I_{1i} \times I_{2j}) = \sum_{i,j} \ell(I_{1i})\ell(I_{2j})$

$$= (b_{11} - a_{11}) \sum_j \ell(I_{2j}) + (b_{12} - a_{12}) \sum_j \ell(I_{2j}) + \cdots + (b_{1n} - a_{1n}) \sum_j \ell(I_{2j}).$$

$$\geq (b_{11} - a_1) \sum_j \ell(I_{2j}) + (b_{12} - b_{11}) \sum_j \ell(I_{2j}) + \cdots + (b_1 - b_{n-1}) \sum_j \ell(I_{2j}).$$

$$\geq (b_1 - a_1) \sum_j \ell(I_{2j}) \geq (b_1 - a_1)(b_2 - a_2) = A.$$

$r(\bigcup_{i,j} \bar{R}_{ij}) \geq A$, hence $\sum_{i,j} r(R_{ij}) \geq A$.

But $\sum_{i,j} r(\bar{R}_{ij}) \leq m_2(R) + \varepsilon$ for an arbitrary $\varepsilon > 0$.

$\therefore$ $m_2(R) \geq A$.

Hence $m_2(R) = A$ for R a compact rectangle.

Assume now that R is open and bounded with area A, then $R = (a,b) \times (c,d)$ where $-\infty < a,b,c,d < \infty$ and (a,b) and (c,d) are open. $R = \bar{R} - \bigcup_{i=1}^{4} L_i$ where $\bar{R} = [a,b] \times [c,d]$ and $\{L_i\}_{i=1}^{4}$ are the four lines forming the boundary of the compact rectangle R. But $m_2(R) = m_2(\bar{R} - \sum_{i=1}^{4} L_i) = m_2(\bar{R}) - \sum_{i=1}^{4} m_2(L_i) = A$, since $\bar{R}$ is compact and $m_2(L_i) = 0$.

Hence for R open or closed, $m_2(R) = A$.

Now, let R be any bounded rectangle, i.e. $R = \{a,b\} \times \{c,d\}$ (where the brackets denote neither open nor closed).

Then $R = R \cup (\bigcup_{i=1}^{4} L_i)$ where $\bar{R}$ is open and bounded and $\bigcup_{i=1}^{4} L_i$ is the boundary, part of which may be open or closed.

(Note: If a side is open, the L_i for that side will be $\emptyset$ and $m_2(\emptyset) = 0$.)

$\therefore$ $m_2(R) = m_2(\bar{R}) + \sum_{i=1}^{4} m_2(L_i) = m_2(\bar{R}) = A$.

Finally, let R be any rectangle, then $\exists$ $\langle R_i \rangle$, a sequence of

bounded rectangles such that $\bigcup R_i = R$.

$\therefore\ m_2(R) = m^2(\bigcup R_i) = \Sigma m_2(R_i)$.

Also, since $R = \bigcup R_i$, Area (R) = Area $(\bigcup R_i) = \Sigma$ Area (R_i)

$= \Sigma\, m_2(R_i)$.

$\therefore$ For an arbitrary rectangle, R, of area A,

$m_2(R) = A$.

Theorem 3.9. $\mathcal{M} \times \mathcal{M}$ is not complete.

Proof. Let $N \subset \mathbb{R}$ such that N is not Lebesgue measurable. But $m_2(\mathbb{R} \times \{0\}) = 0$. hence $m_2(N \times \{0\}) = 0$ since $\mathcal{M}^2$ is complete.

But $N \times \{0\} \notin \mathcal{M} \times \mathcal{M}$ since N is not measurable. That is, if $N \times \{0\} \in \mathcal{M} \times \mathcal{M}$, then every cross-section would be measurable and, in particular, the x-section, N, would be measurable. But N was assumed not to be measurable.

Theorem 3.10. There exists a subset of $\mathbb{R}^2$ which is not Lebesque measurable, i.e. $\exists\ A \subseteq \mathbb{R}^2 \ni A \notin \mathcal{M}^2$.

Proof. Define $\sim$ on $[0,1] \times [0,1]$ by $x \sim y$ if $x - y \in \mathbb{Q}^2$, where $\mathbb{Q}^2 = \{(x,y):\ x,y \in \mathbb{Q}$, the set of rationals$\}$.

By the axiom of choice, let A be a subset of $[0,1] \times [0,1]$ containing exactly one element from each equivalent class.

Define $\oplus$ on $[0,1] \times [0,1]$ by $(x_1,y_1) \oplus (x_2,y_2)$

$$= \begin{cases} (x_1+x_2, y_1+y_2) & \text{if } x_1+x_2<1 \text{ and } y_1+y_2<1. \\ (x_1+x_2, y_1+y_2-1) & \text{if } x_1+x_2<1 \text{ and } y_1+y_2\geq 1. \\ (x_1+x_2-1, y_1+y_2) & \text{if } x_1+x_2\geq 1 \text{ and } y_1+y_2<1. \\ (x_1+x_2-1, y_1+y_2-1) & \text{if } x_1+x_2\geq 1 \text{ and } y_1+y_2\geq 1. \end{cases}$$

For $B \subseteq [0,1] \times [0,1]$ and $x \in [0,1] \times [0,1]$ put $B + x = \{b+x: \ b \in B\}$, and let $\langle r_n \rangle_{n=0}^{\infty}$ be an enumeration of $Q^2 \cap [0,1] \times [0,1]$, with $r_0 = 0$ and $r_n \neq r_m$ for $n \neq m$.

Put $A_n = A \oplus r_n$, for all n.

Then: i) $A_n \cap A_m = \emptyset$ for $n \neq m$.

Assume $A_n \cap A_m \neq \emptyset$, then $\exists$ $x \in A$ and $y \in A \ni x \oplus r_n = y \oplus r_m$.

But this implies $x - y \in Q^2$, hence $x \sim y$.

Since A contains only one representative from each equivalence class, $x = y$.

$\therefore$ $r_n = r_m$ and hence $A_n = A_m$.

ii) If A is measurable, then A_n is measurable and $m_2(A) = m_2(A_n)$, $\forall$ n.

Let $r_n = (s_1, s_2)$. $A = \bigcup_{i=1}^{4} B_i$ where:

$$B_1 = A \cap ([0,1-s_1]\times[0,1-s_2])$$
$$B_2 = A \cap ([0,1-s_1]\times[1-s_2,1])$$
$$B_3 = A \cap ([1-s_1,1]\times[0,1-s_2])$$
$$B_4 = A \cap ([1-s_1,1]\times 1-s_2, 1]).$$

$$A_n = A \oplus r_n = A \oplus (s_1,s_2) = (B_1 \oplus r_n) \cup (B_2 \oplus r_n) \cup$$
$$(B_3 \oplus r_n) \cup (B_4 \oplus r_n) = (B_1 + (s_1,s_2)) \cup (B_2 + (s_1,1-s_2))$$
$$\cup (B_3 + (1-s_1,s_2)) \cup (B_4 + (1-s_1,1-s_2)).$$

Hence if A is measurable, then B_i, $i=1,2,3,4$ is measurable and consequently, A_n is measurable.

$$m_2(A_n) = m_2(B_1 + (s_1,s_2)) + m_2(B_2 + (s_1,1-s_2))$$
$$+ m_2(B_3 + (1-s_1,s_2)) + m_2(B_4 + (1-s_1,1-s_2))$$
$$= m_2(B_1) + m_2(B_2) + m_2(B_3) + m_2(B_4) = m_2(A).$$

$\therefore$ $m_2(A) = m_2(A_n)$, for all n.

iii) Finally, $\bigcup_{n=1}^{\infty} A_n = [0,1] \times [0,1]$.

Let $x \in [0,1] \times [0,1]$. Then $\exists$ $y \in A \ni x \sim y \Rightarrow x-y \in \mathbb{Q}^2$, and thus $x \in A_{n_o}$ for some n_o, i.e. $x \in A \oplus r_{n_o}$ for some $r_{n_o} \in \mathbb{Q}^2$.

Therefore, $m_2(\bigcup_{n=1}^{\infty} A_n) = m_2([0,1] \times [0,1]) = 1$.

But $A_n \cap A_m = \emptyset$ for $n \neq m$ and $m_2(A_n) = m_2(A)$ for all n.

$$\therefore m_2(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m_2(A_n) = \sum_{n=1}^{\infty} m_2(A) = \begin{cases} 0 & \text{if } m_2(A) = 0 \\ \infty & \text{if } m_2(A) > 0. \end{cases}$$

which is a contradiction, since $m_2(\bigcup_{n=1}^{\infty} A_n) = 1$.

$\therefore$ A is not measurable.

The results obtained here apply not only in the plane, but can be found to be the same for n-space. The theory presented here also extends to locally compact spaces and, in particular, to locally compact groups. Although the author has not investigated this aspect, it should lead to many interesting results. Another interesting problem is that of finding relationships between Fubini's Theorem and the Radon-Nikodym Theorem. Investigation into this area should lead to results which provide insight into product measures and the relation of these measures with the measures in the component spaces.

FOOTNOTES

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13. ABSTRACT

In this paper, the essential properties of general Lebesgue outer measure are discussed. The complete measure space, consisting of the general Lebesgue outer measure restricted to the measurable sets, is developed and this measure is shown to be unique. Two characterizations of measurable sets are discussed. The Borel sets are investigated in the plane and more generally, in n-space, and it is shown that the σ-algebra of Borel sets is equal to the product σ-algebra of Borel sets on the line. Finally, the interrelationships between Lebesgue measure in the plane and the product measure of Lebesgue measures on the line are investigated. It is shown that the σ-algebra of Lebesgue measurable sets properly contains the product σ-algebra and that these two measures agree on the product σ-algebra. It is also proven that the σ-algebra of Lebesgue measurable sets is the completion of the product σ-algebra. Examples are provided to illustrate that the product measure spaces discussed are not complete as well as an example of a subset of the plane which is not Lebesgue measurable.

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